Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-11-22

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

We'll split up the problems by the type of integral involved. First we have the ones which integrate a scalar function with respect to $\mathrm{d}s$ on a curve or $\mathrm{d}S$ on a surface. The main method to tackle these problems is just by direct parametrization.

Question 4. The integral is $\int_C (1+y)\,\mathrm{d}s$. The curve C can be parametrized as $\mathbf{r}(t) = \langle t, t^2, 4t^2 + t^4\rangle$, $0 \le t \le 1$. Hence the integral is

$$\int_0^1 (1+t^2)\sqrt{1+(2t)^2+(4t^3)^2}\,\mathrm{d}t.$$

Question 9. We can just parametrize using x, y, i.e. $\mathbf{r}(x,y) = \langle x, y, xy\rangle$, and then we end up with the integral

$$\iint_D \sqrt{y^2+x^2+1}\,\mathrm{d}x\,\mathrm{d}y$$

where D is $x^2+y^2 \le 1$. To get an explicit iterated integral, we'd now have to switch to polar:

$$\int_0^{2\pi}\int_0^1 (\sqrt{r^2+1})r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Alternatively we could've parametrized with polar from the start, via $\mathbf{r}(\theta, r) = \langle r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta\rangle$. I'll let you check that we end up with the same integral.

Next up we have a number of line integrals which compute the work done by a vector field along a curve. For these, our options are

- direct parametrization,
- Stokes' Theorem in 3D, or Green's Theorem in 2D, or
- FTLI.

However, the second bullet point can only be applied when the curve is closed (so that it bounds some 2D region), while the third requires the vector field to be conservative, i.e. of the form ∇f for some scalar function f.

Question 2. It would be somewhat annoying to do this via direct parametrization, as this curve has three parts to consider. However, it is a closed curve: it is the boundary of the region $x^2 \le y \le 1$, $0 \le x \le 1$. It is traced out counterclockwise, so Green's Theorem yields the equivalent double integral:

$$\int_0^1\int_{x^2}^1 (y-2x^2y)\,\mathrm{d}y\,\mathrm{d}x.$$

Question 7. This is similar to the preceding problem, except now it takes place in $\mathbb{R}^3$. The curve C is the boundary of a filled-in ellipse S on the plane $x+y+z=1$, with $x^2+y^2 \le 9$. Since C is oriented counterclockwise when viewed from above, S should be oriented upwards to have $\partial S = C$. Stokes' Theorem then says that our integral is equal to

$$\iint_S \langle 0, x^2, y^2\rangle \cdot \mathrm{d}\mathbf{S}.$$

Now we can parametrize S, e.g. as $\mathbf{r}(x,y) = \langle x, y, 1-x-y\rangle$.

$$\iint_{x^2+y^2\le 9} \langle 0, x^2, y^2\rangle \cdot \langle 1,1,1\rangle\,\mathrm{d}x\,\mathrm{d}y$$

where $\langle 1,1,1\rangle = \mathbf{r}_x \times \mathbf{r}_y$ points upwards, as we want. Then we can switch to polar:

$$\int_0^{2\pi}\int_0^2 r^3\,\mathrm{d}r\,\mathrm{d}\theta.$$

This problem could've also been approached via direct parametrization, but one would have to deal with some annoyingly high powers of trigonometric functions in that case.

Question 8. The vector field isn't conservative and C is not a closed curve. Both of these issues are not easy to fix in this problem, so it's a good idea to just directly parametrize. The line segment can be parametrized as $\mathbf{r}(t) = \langle 1+3t, t, 2t\rangle$, $0 \le t \le 1$, and then we rewrite the integral as

$$\int_0^1 \langle 4t^2, (1+3t)^2, t^2\rangle \cdot \langle 3,1,2\rangle \,\mathrm{d}t = \int_0^1 (12t^2 + (1+3t)^2 + 2t^2)\,\mathrm{d}t.$$

Question 10. This is a famous vector field that I've done examples with a few times in class. If you wanted to parametrize C, the most direct way to do it would be as

$$x = (2+\cos(4\theta))\cos\theta$$
$$y = (2+\cos(4\theta))\sin\theta.$$

Since the worksheet doesn't ask you to evaluate the integral after setting it up, you could write down a (rather ugly) single integral using this parametrization and call it a day.

However, the main conceptual point of this problem is that the vector field has zero curl. It is *not* conservative on its entire domain, but if C' denotes a circle of radius $a < 1$ centered at $(0,0)$ traced out counterclockwise, then applying Green's Theorem to the 2D region D between C' and C yields the equation

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} - \int_{C'} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \nabla \times \mathbf{F})\cdot \mathbf{k}\,\mathrm{d}x\,\mathrm{d}y = \iint_D 0\,\mathrm{d}x\,\mathrm{d}y = 0$$

so $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_{C'} \mathbf{F}\cdot\mathrm{d}\mathbf{r}$. C' is very easy to parametrize: $x = a\cos t, y = a\sin t, 0 \le t \le 2\pi$. Note that a is a fixed constant, not a parameter. (The $a < 1$ condition is just to ensure that C' lies entirely inside of C so that the region D makes sense.) With this parametrization,

$$\int_{C'} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \langle -\frac{1}{a}\sin t, \frac{1}{a}\cos t\rangle \cdot \langle -a\sin t, a\cos t\rangle\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t.$$

Question 11. In this problem, C is again a closed curve, now in 3D. So there's a good chance we ought to apply Stokes' Theorem. However, to do so, we must find a suitable surface S such that $\partial S = C$. No such surface S is obvious from the description of C.

Let's compute the curl of the vector field (since we'll need to do that anyway if we want to apply Stokes' Theorem). We find

$$\nabla \times \langle -xy^2, x^2y, e^{5z}\rangle = \langle 0, 0, 4xy\rangle.$$

This vector field has only nonzero z-component, which means that its flux through any vertical surface is zero. We use this as a guide for choosing an "easy" surface S: let's take the portion of the cylinder $x^2+y^2 = 1$ between the curve C and $z = -10$ (totally arbitrary number), together with the bottom lid $z = -10, x^2+y^2 \le 1$. Since C is oriented counterclockwise when viewed from above, we orient the cylindrical part of S inwards, and the bottom part of S upwards, so that $\partial S = C$. So our integral is equal to

$$\iint_S \langle 0,0,4xy\rangle \cdot \mathrm{d}\mathbf{S}$$

and since the flux is zero through the vertical cylindrical side, this is just the flux through the bottom, which we can parametrize using just $\mathbf{r}(x,y) = \langle x, y, -10\rangle$ for example:

$$\iint_{x^2+y^2\le 1} 4xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 (4\cos\theta\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Next we have a handful of flux integrals through surfaces. Our techniques for dealing with these:

- direct parametrization,
- Stokes' Theorem, or
- the Divergence Theorem.

Note that the second approach is applicable only when the integrand has the form $\nabla \times \mathbf{F}$ for some vector field $\mathbf{F}$, and the third approach requires the surface to be closed, i.e. it must bound some solid 3D region.

Question 1. The integrand isn't written in the form $\nabla \times$ (vector field), so we can't apply Stokes' Theorem[1]. The surface isn't closed either, and fixing that to apply the Divergence Theorem seems like way more trouble than it's worth. So we'll just have to use a direct parametrization, say

$$\mathbf{r}(\theta, r) = \langle r\cos\theta, r\sin\theta, r\rangle$$

with $0 \le \theta \le 2\pi$ and $1 \le r \le 3$. I'll omit writing down the integral, but you should be very comfortable with finishing the problem at this point.

Question 3. The problem presents the surface S to you as a boundary! That's a good sign to use the Divergence Theorem, especially considering that you'd have to do three separate surface integrals if you were to tackle the problem directly. With the Divergence Theorem though, this just becomes

$$\iiint_E 2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where E is the 3D region $x^2 + z^2 \le 1, y \ge 0, x + y \le 2$. To set up this integral, one would probably do $\mathrm{d}y$ first, and then "polar" in the xz-plane, e.g. $x = r\cos\theta, z = r\sin\theta$. (One could also describe this as modified cylindrical coordinates.)

$$\int_0^{2\pi} \int_0^1 \int_0^{2-r\cos\theta} 2r \, \mathrm{d}y \, \mathrm{d}r \, \mathrm{d}\theta.$$

Question 5. Since this integral is the flux of a curl of a vector field, it strongly suggests the use of Stokes' Theorem. To apply it, we must identify ∂S, which is described by $4x^2 + y^2 + 4z^2 = 4, y = 0$, so equivalently $x^2 + z^2 = 4, y = 0$. This is a circle in the xz-plane. We need to parametrize it so that its orientation is consistent with that of S, so $x = \sin t, y = 0, z = \cos t$ will do (draw a picture and check this with the RHR!). Then we can apply Stokes':

$$\int_0^{2\pi} \langle 1, e^{\sin t \cos t}, (\sin t)^2 \cos t\rangle \cdot \langle \cos t, 0, -\sin t\rangle \, \mathrm{d}t = \int_0^{2\pi} (\cos t - (\sin t)^3 \cos t) \, \mathrm{d}t.$$

Question 6. Although the ellipsoid isn't impossible to parametrize directly, it's a closed surface and it's much nicer to use the Divergence Theorem to solve this problem, which converts the integral to

$$-\iiint_E (1 + x) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where E is the filled-in ellipsoid $x^2 + 2y^2 + 3z^2 \le 4$. Note the negative sign, because the starting surface was oriented inwards, rather than outwards. Now we apply the change of variables

$$x = u$$
$$y = v/\sqrt{2}$$
$$z = w/\sqrt{3}$$

so that the region becomes $u^2 + v^2 + w^2 \le 4$ in uvw-space, which is a filled-in sphere. The integral is now

$$-\iiint_{u^2+v^2+w^2\le 4} (1 + u)\frac{1}{\sqrt{6}} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w$$

which we can write in spherical as

$$-\int_0^{2\pi} \int_0^{\pi} \int_0^2 (1 + \rho\cos\phi)\frac{1}{\sqrt{6}}\rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

[1] It can't be written in that form either, since the divergence of $\langle -x, -y, z^3\rangle$ is nonzero. But in all my time teaching Math 53, I haven't seen a problem where students were expected to find $\mathbf{F}$ given $\nabla \times \mathbf{F}$, so it's safe to say that you should only use Stokes' on a flux integral when the problem explicitly presents the integrand to you as a curl.

The last question on the worksheet is a bit of an outlier. Just as how Green's Theorem can be used to convert double integrals in the plane to work line integrals of appropriately chosen vector fields, the Divergence Theorem can be used to convert triple integrals in space to flux surface integrals of appropriately chosen vector fields. This is typically useful when the 3D region to be integrated over is difficult to describe, but has a boundary surface that is given parametrically instead.

Question 12. For starters, the curve in the xy-plane can be parametrized as

$$\begin{aligned} x &= \cos(2\theta)\cos\theta \\ y &= \cos(2\theta)\sin\theta, \end{aligned}$$

$-\pi/4 \le \theta \le \pi/4$. When we rotate this around the y-axis, we introduce a new "rotation parameter" α. Note that since we are rotating around the y-axis, the expression for y will be unchanged. The expressions for x, z will be the familiar ones for a circle of radius $\cos(2\theta)\cos\theta$, with α as the circle's rotation parameter:

$$\begin{aligned} x &= \cos(2\theta)\cos\theta\cos\alpha \\ y &= \cos(2\theta)\sin\theta \\ z &= \cos(2\theta)\cos\theta\sin\alpha, \end{aligned}$$

$-\pi/4 \le \theta \le \pi/4$ and $0 \le \alpha \le 2\pi$. This is our parametrization of the surface ∂E. The Divergence Theorem says

$$\iiint_E 1\,\mathrm{d}V = \iint_{\partial E} \mathbf{F}\cdot\mathbf{dS}$$

as long as $\nabla\cdot\mathbf{F} = 1$. So we have a lot of freedom in choosing $\mathbf{F}$. The obvious choices are $\langle x, 0, 0\rangle$, $\langle 0, y, 0\rangle$, and $\langle 0, 0, z\rangle$. But there's no need to be hasty and decide just yet—let's examine $\mathbf{dS}$ first. Since we need ∂E to be oriented outwards, the RHR tells us that $\mathrm{d}\mathbf{S} = \mathbf{r}_\theta \times \mathbf{r}_\alpha\,\mathrm{d}\theta\,\mathrm{d}\alpha$.

$$\mathbf{r}_\theta \times \mathbf{r}_\alpha = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta)\cos\alpha & -2\sin(2\theta)\sin\theta & (-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta)\sin\alpha \\ -\cos(2\theta)\cos\theta\sin\alpha & \cos(2\theta)\cos\theta & \cos(2\theta)\cos\theta\cos\alpha \end{bmatrix}$$

Let's take $\mathbf{F} = \langle 0, y, 0\rangle$, because

- the expression for y itself in terms of the parameters is not so bad, and
- the y-component of $\mathbf{r}_\theta \times \mathbf{r}_\alpha$ is relatively nice too, as the αs will go away and just leave us

$$(-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta)\cos(2\theta)\cos\theta.$$

Note that this is the only component of the cross product that will matter if we take $\mathbf{F} = \langle 0, y, 0\rangle$.

We end up with the integral

$$\int_{-\pi/4}^{\pi/4}\int_0^{2\pi} \frac{1}{2}\cos^2(2\theta)\sin(2\theta)(-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta)\,\mathrm{d}\theta\,\mathrm{d}\alpha.$$

This was a (very hard) past final exam question. On that exam, the values of some trigonometric integrals were provided, so that the problem was essentially done once you wrote down this last integral.